Let T be the equivalence relation on $\mathbf{Z}^+ \times \mathbf{Z}^+$ defined by $(a, b)T(c, d) \Leftrightarrow 3 \mid (ab - cd)$. **SCORE: _____ / 14 PTS**

<u>You do not need to prove that T is an equivalence relation.</u>

[a] Find 3 elements of the form $(1, b)$ such that $(1, b)T(1, 1)$.

① IF YOU GOT AT LEAST 2 PAIRS

+① FOR A 3RD PAIR

$$(1, b)T(1, 1) \Leftrightarrow 3 \mid (b - 1) \qquad (1, 1), (1, 4), (1, 7), (1, 10), (1, 13)$$

ANY 3 OF THESE ARE OK

[b] Find 3 elements of the form $(a, 2)$ such that $(a, 2)T(1, 1)$.

$$(a, 2)T(1, 1) \Leftrightarrow 3 \mid (2a - 1) \qquad (2, 2), (5, 2), (8, 2), (11, 2), (14, 2)$$

ANY 3 OF THESE ARE OK

① IF YOU GOT AT LEAST 2 PAIRS

+① FOR A 3RD PAIR

[c] Find 5 elements of the equivalence class containing $(1, 1)$ such that none of the elements have the form $(1, b), (b, 1), (a, 2)$ nor $(2, a)$.

$$(a, b)T(1, 1) \Leftrightarrow 3 \mid (ab - 1) \qquad (4, 1), (4, 4), (4, 7), (5, 2), (5, 5)$$

OTHER ANSWERS POSSIBLE

① FOR EACH PAIR

[d] Describe the partition induced on $\mathbf{Z}^+ \times \mathbf{Z}^+$ by T.
Your answer should not involve divisibility or any definition of divisibility.

there are 3 equivalence classes
determined by the remainder when the product of the two elements in an ordered pair is divided by 3

5

one class contains all ordered pairs whose product is a multiple of 3
one class contains all ordered pairs whose product is 1 more than a multiple of 3
one class contains all ordered pairs whose product is 2 more than a multiple of 3

One of the following relations is an equivalence relation, and the other is not. **SCORE: _____ / 21 PTS**

[a] relation R on $\mathbf{Z} \times \mathbf{Z}$ defined by $(a, b)R(c, d)$ if and only if $ad = bc$

[b] relation S on $\mathbf{Z}^*$ defined by xSy if and only if $y = x \cdot 2^n$ for some $n \in \mathbf{Z}$

[i] Which one is not an equivalence relation ? **<u>Justify your answer clearly & briefly</u>.**

3 R is not an equivalence relation since $1 \cdot 0 = 2 \cdot 0$ (ie. $(1, 2)R(0, 0)$) and $0 \cdot 4 = 0 \cdot 3$ (ie. $(0, 0)R(3, 4)$) but $1 \cdot 4 \neq 2 \cdot 3$, so R is not transitive

[ii] Justify that the other relation is an equivalence relation by proving informally (as shown in lecture) that it satisfies the definition **<u>(which includes writing out symbolically exactly what you are proving)</u>**.

reflexive:

2 $\forall x \in \mathbf{Z}^*, x = x \cdot 2^n$ for some $n \in \mathbf{Z}$

1 $x = x \cdot 2^0$

symmetric:

2 $\forall x, y \in \mathbf{Z}^*, (y = x \cdot 2^n$ for some $n \in \mathbf{Z} \rightarrow x = y \cdot 2^k$ for some $k \in \mathbf{Z})$

$y = x \cdot 2^n \rightarrow x = y \cdot 2^{-n}$ and $-n \in \mathbf{Z}$ 1

1

transitive:

2 $\forall x, y, z \in \mathbf{Z}^*, (y = x \cdot 2^n$ for some $n \in \mathbf{Z} \wedge z = y \cdot 2^k$ for some $k \in \mathbf{Z} \rightarrow z = x \cdot 2^p$ for some $p \in \mathbf{Z})$

$y = x \cdot 2^n \wedge z = y \cdot 2^k \rightarrow z = x \cdot 2^n \cdot 2^k = x \cdot 2^{n+k}$ and $n + k \in \mathbf{Z}$ 1

1

[iii] If R is the equivalence relation, find 5 elements of the equivalence class containing $(4, 5)$.
If S is the equivalence relation, find 5 elements of the equivalence class containing 3.

$3Ry \Leftrightarrow y = 3 \cdot 2^n$ for some $n \in \mathbf{Z}$ 3, 6, 12, 24, 48 2

[iv] Describe the partition induced by the equivalence relation.

1 there is one equivalence class that only contains 0

2 each odd number has a corresponding equivalence class that contains that odd number

2 along with that odd number multiplied by any (positive integer) number of 2's